

Towards an Axiomatic Formulation of Noncommutative Quantum Field Theory

**M. Chaichian[†], M. N. Mnatsakanova[§], K. Nishijima[‡], A. Tureanu[†]
and Yu. S. Vernov[¶]**

[†]*Department of Physics, University of Helsinki and Helsinki Institute of Physics,
P.O. Box 64, FIN-00014 Helsinki, Finland*

[§] *Skobeltsyn Institute of Nuclear Physics, Moscow State University, 119992, Vorobyevy Gory,
Moscow, Russia*

[‡]*Department of Physics, University of Tokyo 7-3-1 Hongo, Bunkyo-ku,
Tokyo 113-0033, Japan*

[¶]*Institute for Nuclear Research, Russian Academy of Sciences,
60th October Anniversary prospect 7a, 117312, Moscow, Russia*

Abstract

We propose new Wightman functions as vacuum expectation values of products of field operators in the noncommutative space-time. These Wightman functions involve the \star -product among the fields, compatible with the twisted Poincaré symmetry of the noncommutative quantum field theory (NC QFT). In the case of only space-space noncommutativity ($\theta_{0i} = 0$), we prove the CPT theorem using the noncommutative form of the Wightman functions. As a byproduct, we arrive at the general conclusion of the following theorem: the violation of CPT invariance implies the violation of not only Lorentz invariance, but also its subgroup of symmetry, $O(1, 1) \times SO(2)$. We also show that the spin-statistics theorem, demonstrated for the simplest case of a scalar field, holds in NC QFT within this formalism.

1 Introduction

The axiomatic approach to quantum field theory (QFT) built up by Wightman, Jost, Bogoliubov, Haag and others made QFT a consistent, rigorous theory (for references, see [1]-[4]). In the framework of this approach, fundamental results, such as the CPT and spin-statistics theorems, were proven in general, without any reference to a specific theory and a Lagrangian or Hamiltonian. In addition, the axiomatic formulation of QFT has given the possibility to derive analytical properties of scattering amplitudes and, as a result, dispersion relations. Consequently, various rigorous bounds on the high-energy behaviour of scattering amplitudes were obtained [5].

At present, noncommutative quantum field theory (NC QFT) attracts a great deal of attention. The study of such theories has received a considerable impetus after it was shown that they appear naturally, in some cases, as low-energy limit of open string theory in the presence of a constant antisymmetric background field [6]. In this context, the coordinate operators of a space-time satisfy the Heisenberg-like commutation relations

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu}, \quad (1.1)$$

where $\theta_{\mu\nu}$ is a constant antisymmetric matrix of dimension (length)². For the study of NC QFT it is customary to define for the field operators on the noncommutative space-time, $\phi(\hat{x})$, their Weyl symbols, $\phi(x)$, whose algebra is isomorphic to the initial operator algebra. The connection between the operators $\phi(\hat{x})$ and their Weyl symbols $\phi(x)$ is achieved through the Weyl-Moyal correspondence, which requires that products of operators are replaced by Moyal \star -products of their Weyl symbols:

$$\phi(\hat{x})\psi(\hat{x}) \rightarrow \phi(x) \star \psi(x), \quad (1.2)$$

where the Moyal \star -product is defined as

$$(\phi \star \psi)(x) = \phi(x) e^{\frac{i}{2}\theta^{\mu\nu}\overleftrightarrow{\partial_\mu}\overleftrightarrow{\partial_\nu}} \psi(x). \quad (1.3)$$

In the following, by field operators we shall understand such Weyl symbols. Since the matrix $\theta^{\mu\nu}$ is constant and does not transform under Lorentz transformations, the Lorentz symmetry $SO(1, 3)$ is broken, while the translational invariance is preserved.

We shall consider throughout this paper only the case of space-space noncommutativity, i.e. $\theta_{0i} = 0$, since theories with $\theta_{0i} \neq 0$ cannot be obtained as low-energy limit from string theory [6]. Besides, it has been shown that such field theories violate perturbative unitarity [7] and causality [8, 9].

The study of NC QFT has been mostly done in the Lagrangian approach (for reviews, see [10, 11]). However, it would be of importance to develop also an axiomatic formulation of NC QFT, which does not refer to a specific Lagrangian.

The first step in this direction was made in [12], where the usual Wightman functions, defined as

$$W(x_1, x_2, \dots, x_n) = \langle 0 | \phi(x_1)\phi(x_2)\dots\phi(x_n) | 0 \rangle \quad (1.4)$$

were investigated, but based on the symmetry group $O(1, 1) \times SO(2)$, which is the residual space-time symmetry of NC space-time with $\theta_{0i} = 0$. Using the usual Wightman functions as in (1.4), the validity of the CPT theorem was shown in [12].

Perhaps the most serious problem with NC QFT treated in the residual symmetry approach, used to be the fact that the representations of the fields did not match the actual symmetry of the theory. The groups $O(1, 1)$ and $SO(2)$ are both Abelian, having only one-dimensional irreducible representation, and therefore not supporting the concept of spin, which is essential for particle physics. A rigorous axiomatic approach should include the proof of the spin-statistics theorem, but the spin simply does not exist in a $O(1, 1) \times SO(2)$ -invariant theory, as pointed out in [12]. The solution to the representations problem was the main physical implication of the uncovering of the twisted Poincaré symmetry of noncommutative quantum field theory with Heisenberg-like commutation relations of the coordinates [13]. Twisted Poincaré symmetry proved to be the new concept of relativistic invariance for NC QFT [14] and it shaped the field in a more rigorous way. Based on the twisted Poincaré symmetry and its various consequences, we are now in a position to formulate a well-argumented axiomatic approach to NC QFT.

The role of twisted Poincaré symmetry in this paper is two-fold. Firstly, it is needed in order to justify the use of the star-product between functions in two different space-time points, since in this case the newly defined Wightman functions will have the same symmetry as the one of space-time commutation relation. In particular, for scalar fields the new Wightman functions will be explicitly scalars under the twisted Poincaré transformations. Secondly, since the Lorentz invariance is violated in such NC theories down to the product

of two Abelian groups, one has not the concept of spin and thus cannot speak of the spin-statistics theorem altogether, unless one invokes the existence of twisted Poincaré thanks to its representation theory being identical with the one of the usual Poincaré symmetry.

The existence of the class of test functions in the case of NC quantum field theories (as done in Ref. [15]), is crucial for utilizing the analytical properties of the smeared new Wightman functions needed for the rigorous proof of the CPT and spin-statistics theorems, as it is the case also for the usual commutative quantum field theories.

In this paper, we shall formulate the axiomatic approach to NC QFT mainly guided by the twisted Poincaré symmetry. The same symmetry arguments impose also the adoption of a new form for the Wightman functions. On the ground of this coherent formulation we shall prove the CPT theorem for theories with space-space noncommutativity and also give the proof of the spin-statistics theorem for the case of a spinless field, for simplicity.

2 Axiomatic approach to NC QFT

2.1 Twisted Poincaré symmetry

Since the twisted Poincaré symmetry [13, 14] is our guiding line in this formulation, we shall review a few main concepts and formulas, for the consistency of the argumentation.

The twisted Poincaré algebra is the universal enveloping of the Poincaré algebra $\mathcal{U}(\mathcal{P})$, viewed as a Hopf algebra, deformed with the Abelian twist element [16] (see also the monographs [17])

$$\mathcal{F} = \exp\left(\frac{i}{2}\theta^{\mu\nu}P_\mu \otimes P_\nu\right), \quad (2.5)$$

where $\theta_{\mu\nu}$ is a constant antisymmetric matrix (it does not transform under the Lorentz transformations) and P_μ are the translation generators. This induces on the algebra of representations of the Poincaré algebra the deformed multiplication,

$$m \circ (\phi \otimes \psi) = \phi\psi \rightarrow m_\star \circ (\phi \otimes \psi) = m \circ \mathcal{F}^{-1}(\phi \otimes \psi) \equiv \phi \star \psi, \quad (2.6)$$

which is precisely the well-known Weyl-Moyal \star -product (taking the Minkowski space realization of P_μ , i.e. $P_\mu = -i\partial_\mu$):

$$\star = \exp\left(\frac{i}{2}\theta^{\mu\nu}\overleftarrow{\partial}_\mu \overrightarrow{\partial}_\nu\right). \quad (2.7)$$

In particular, taking in (2.6) $\phi(x) = x_\mu$ and $\psi(x) = x_\nu$, one obtains:

$$[x_\mu, x_\nu]_\star = i\theta_{\mu\nu}. \quad (2.8)$$

This is the usual commutation relation of the Weyl symbols of the noncommuting coordinate operators \hat{x} , (2.8), which is obtained in the Weyl-Moyal correspondence.

The twist (2.5) does not affect the actual commutation relations of the generators of the Poincaré algebra \mathcal{P} :

$$\begin{aligned} [P_\mu, P_\nu] &= 0, & [M_{\mu\nu}, P_\alpha] &= -i(\eta_{\mu\alpha}P_\nu - \eta_{\nu\alpha}P_\mu), \\ [M_{\mu\nu}, M_{\alpha\beta}] &= -i(\eta_{\mu\alpha}M_{\nu\beta} - \eta_{\mu\beta}M_{\nu\alpha} - \eta_{\nu\alpha}M_{\mu\beta} + \eta_{\nu\beta}M_{\mu\alpha}). \end{aligned} \quad (2.9)$$

Consequently also the Casimir operators remain the same and the representations and classifications of *particle states* are identical to those of the ordinary Poincaré algebra. The

question of the fields, constructed by the method of induced representations, is more subtle [18, 19], but crucial for the edification of the axiomatic formulation on NC QFT, and we shall review it in the next subsection.

The twist deforms the action of the generators in the tensor product of representations – the so-called *coprod*uct. In the case of the usual Poincaré algebra, the coproduct $\Delta_0 \in \mathcal{U}(\mathcal{P}) \times \mathcal{U}(\mathcal{P})$ is symmetric (the usual Leibniz rule),

$$\Delta_0(Y) = Y \otimes 1 + 1 \otimes Y, \quad (2.10)$$

for all the generators $Y \in \mathcal{P}$. The twist \mathcal{F} deforms the coproduct Δ_0 to $\Delta_t \in \mathcal{U}_t(\mathcal{P}) \times \mathcal{U}_t(\mathcal{P})$ as:

$$\Delta_0(Y) \mapsto \Delta_t(Y) = \mathcal{F}\Delta_0(Y)\mathcal{F}^{-1}. \quad (2.11)$$

This similarity transformation is compatible with all the properties of $\mathcal{U}(\mathcal{P})$ as a Hopf algebra, since \mathcal{F} satisfies the twist equation [16]:

$$\mathcal{F}_{12}(\Delta_0 \otimes id)\mathcal{F} = \mathcal{F}_{23}(id \otimes \Delta_0)\mathcal{F}, \quad (2.12)$$

where $\mathcal{F}_{12} = \mathcal{F} \otimes 1$ and $\mathcal{F}_{23} = 1 \otimes \mathcal{F}$. The eq. (2.12) ensures the associativity of the \star -product (2.6). This is an important point, to which we shall return when discussing the new form of the Wightman functions.

The twisted coproducts of the generators of Poincaré algebra turn out to be:

$$\Delta_t(P_\mu) = \Delta_0(P_\mu) = P_\mu \otimes 1 + 1 \otimes P_\mu, \quad (2.13)$$

$$\begin{aligned} \Delta_t(M_{\mu\nu}) &= M_{\mu\nu} \otimes 1 + 1 \otimes M_{\mu\nu} \\ &- \frac{1}{2}\theta^{\alpha\beta}[(\eta_{\alpha\mu}P_\nu - \eta_{\alpha\nu}P_\mu) \otimes P_\beta + P_\alpha \otimes (\eta_{\beta\mu}P_\nu - \eta_{\beta\nu}P_\mu)]. \end{aligned} \quad (2.14)$$

Thus the twisted coproduct of the momentum generators is identical to the primitive coproduct, eq. (2.13), meaning that translational invariance is preserved, while the twisted coproduct of the Lorentz algebra generators, eq. (2.14), is nontrivial, implying the violation of Lorentz symmetry.

It is essential for our purpose to note that, by fixing conveniently the frame of reference, the matrix $\theta_{\mu\nu}$ takes a block diagonal form:

$$\theta_{\mu\nu} = \begin{pmatrix} 0 & \theta' & 0 & 0 \\ -\theta' & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta \\ 0 & 0 & -\theta & 0 \end{pmatrix}. \quad (2.15)$$

This form emphasizes the stability group of the matrix $\theta_{\mu\nu}$, i.e. $SO(1, 1) \times SO(2)$, which becomes $O(1, 1) \times SO(2)$ as soon as $\theta' = 0$ (space-space noncommutativity). One immediately notices that the coproducts of the generators of $SO(1, 1)$ and $SO(2)$, i.e. M_{01} and M_{23} , remain primitive in this frame of reference: $\Delta_t(M_{01}) = \Delta_0(M_{01})$ and $\Delta_t(M_{23}) = \Delta_0(M_{23})$, which is the mark of the preservation of ordinary invariance under the corresponding Lorentz transformation.

Although the formulation of the symmetry as twisted Poincaré algebra is very useful for noting the solution to the representation problem, it is important for physical application to understand how the corresponding finite transformations act (see, e.g., [18]). In the case of the twisted Poincaré algebra $\mathcal{U}_t(\mathcal{P})$, its dual is the algebra of function $F_\theta(G)$ on the Poincaré

group G with deformed multiplication. The algebra $F(G)$, dual to ordinary $\mathcal{U}(\mathcal{P})$, is generated by the elements Λ_ν^μ and \mathbf{a}^μ , which are complex-valued functions, such that when applied to suitable elements of the Poincaré group, they would return the familiar real-valued entries of the matrix of finite Lorentz transformations, Λ_ν^μ , or the real-valued parameters of finite translations, a^μ . In the case of $F_\theta(G)$, the functions \mathbf{a}^μ are no more complex-valued and the following commutation relations are obtained by requiring the duality between $\mathcal{U}_t(\mathcal{P})$ and $F_\theta(G)$:

$$\begin{aligned} [\mathbf{a}^\mu, \mathbf{a}^\nu] &= i\theta^{\mu\nu} - i\Lambda_\alpha^\mu \Lambda_\beta^\nu \theta^{\alpha\beta}, \\ [\Lambda_\nu^\mu, \mathbf{a}^\alpha] &= [\Lambda_\alpha^\mu, \Lambda_\beta^\nu] = 0, \quad \Lambda_\alpha^\mu, \mathbf{a}^\mu \in F_\theta(G). \end{aligned} \quad (2.16)$$

Again, one notices that, when applied to elements of the Lorentz group which do not belong to the above mentioned $SO(1, 1)$ or $SO(2)$ stability subgroups, the eqs. (2.16) lead to non-commutative translations, whose physical interpretation is elusive [19]. Ordinarily, in $F(G)$, the functions \mathbf{a}^μ applied to elements of Lorentz group vanish, but in the case of $F_\theta(G)$ this peculiar result obstructs the physical interpretation of finite twisted Poincaré transformations of even the coordinates, and the problem becomes worse for the transformation of fields.

It turns out that these problems are solved by the same stratagem which will permit us to define noncommutative fields using the method of induced representations.

2.2 Noncommutative field operators

One would be tempted to say that the construction of a NC quantum field theory through the Weyl-Moyal correspondence is equivalent to the procedure of redefining the multiplication of functions, so that it is consistent with the twisted coproduct of the Poincaré generators (2.11) [13].

However, the definition of noncommutative fields and the action of the twisted Poincaré transformations on them is not a trivial one. Ordinary relativistic fields are defined by the method of induced representations. In the commutative setting, Minkowski space is realized as the quotient of the Poincaré group by the Lorentz group, and a classical field is a section of a vector bundle induced by some representation of the Lorentz group. This construction does not generalize to the noncommutative case, because the universal enveloping algebra of the Lorentz Lie algebra is not a Hopf subalgebra of the twisted Poincaré algebra. As a result, Minkowski space $\mathbb{R}^{1,3}$, which in the commutative setting is realized as the quotient of the Poincaré group G by the Lorentz group L , G/L , has no noncommutative analogue.

This can be intuitively seen if we recall that an ordinary commutative field is typically written as

$$\Phi = f \otimes v, \quad f \in C^\infty(\mathbb{R}^{1,3}), \quad v \in V,$$

where $C^\infty(\mathbb{R}^{1,3})$ is the set of smooth functions on Minkowski space and V is a Lorentz-module. The tensor product requires that the action of the Lorentz generators on Φ be taken with twisted coproduct. However, this implies an action of the momentum generators P_μ on the representations of the Lorentz group v , and such an action is simply not defined.

One proposal for bypassing this predicament is to consider V a Poincaré-module, with trivial action of the momentum generators [18]. Another proposal – which we shall adopt in this paper – is to maintain V as a Lorentz module, but to forbid the transformations which cannot go through [19]. In this way, we induce only the Lorentz transformations corresponding to the stability group of $\theta_{\mu\nu}$, but the fields will carry representations of the full Lorentz group; consequently, the particle spectrum of the noncommutative quantum field

theory with twisted Poincaré symmetry will have the richness of the relativistic quantum field theory. At the same time, the problem of noncommutative finite translations is also solved.

We emphasize that the fact that only certain Lorentz transformations are allowed on the noncommutative fields is a strong indication of the Lorentz symmetry violation. The differences between ordinary and noncommutative quantum fields are drastic and there is no way to justify, based on the twisted Poincaré symmetry, the claim that the noncommutative fields transform under all Lorentz transformations as ordinary relativistic fields [21].

2.3 Noncommutative Wightman functions

In the ordinary axiomatic field theory, Wightman functions are defined as vacuum expectation values of products of relativistic field operators,

$$W(x_1, x_2, \dots, x_n) = \langle 0 | \phi(x_1) \phi(x_2) \dots \phi(x_n) | 0 \rangle. \quad (2.17)$$

The twist changes the multiplication in the algebra of representation of the Poincaré algebra, and we expect that this should apply as well to Wightman functions. Indeed, the product of field operators with independent arguments, $\phi(x_1)$ and $\phi(x_2)$, in as far as the space-time dependence is concerned, is an element of the tensor product of two copies of $C_\theta^\infty(\mathbb{R}^{1,3})$, and the rule of multiplication in this case is:

$$\begin{aligned} (f_1 \otimes 1)(1 \otimes f_2) &= f_1 \otimes f_2, \\ (1 \otimes f_2)(f_1 \otimes 1) &= (\mathcal{R}_2 f_1) \otimes (\mathcal{R}_1 f_2), \quad f_1, f_2 \in C_\theta^\infty(\mathbb{R}^{1,3}), \end{aligned} \quad (2.18)$$

where $\mathcal{R} \in \mathcal{U}(\mathcal{P}) \otimes \mathcal{U}(\mathcal{P})$ is the universal \mathcal{R} -matrix, which relates by a similarity transformation the twisted coproduct Δ_t and its opposite $\Delta_t^{op} = \sigma \circ \Delta_t$,

$$\mathcal{R} \Delta_t = \Delta_t^{op} \mathcal{R}. \quad (2.19)$$

In the case of twisted Poincaré algebra with the twist (2.5), the expression of the \mathcal{R} -matrix is

$$\mathcal{R} = \mathcal{F}_{21} \mathcal{F}^{-1} = \mathcal{F}^{-2}. \quad (2.20)$$

The property (2.18) is encoded in the product of functions with independent arguments as an extension of the \star -product:

$$f(x) \star g(y) = f(x) e^{\frac{i}{2} \theta^{\mu\nu} \frac{\vec{\partial}}{\partial x^\mu} \frac{\vec{\partial}}{\partial y^\nu}} g(y). \quad (2.21)$$

Such a generalization of the \star -product for noncoinciding space-time points has been previously proposed and used in a different context [11].

This generalization can be also motivated by the fact that in the commutation relation of coordinate operators,

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu},$$

the labels by which we designate the coordinate operators are not relevant, since the non-commutativity is only between different components of space directions (see [20]).

In this formulation, we propose for the noncommutative Wightman functions the expression:

$$W_\star(x_1, x_2, \dots, x_n) = \langle 0 | \phi_\theta(x_1) \star \phi_\theta(x_2) \star \dots \star \phi_\theta(x_n) | 0 \rangle. \quad (2.22)$$

In this expression and in the following, we shall denote the noncommutative field operators by the subscript θ , in order to emphasize once more their different properties from the ordinary relativistic field operators. The \star -product of operators in (2.22)

$$\phi_\theta(x_1) \star \phi_\theta(x_2) \star \dots \star \phi_\theta(x_n) = e^{\frac{i}{2}\theta^{\mu\nu} \sum_{a < b} \frac{\partial}{\partial x_a^\mu} \frac{\partial}{\partial x_b^\nu}} \phi_\theta(x_1) \phi_\theta(x_2) \dots \phi_\theta(x_n), \quad (2.23)$$

is obviously associative and for the coinciding points $x_1 = x_2 = \dots = x_n$ becomes identical to the multiple Moyal \star -product.

2.4 Translational invariance of the Wightman functions

The NC Wightman functions (2.22) are translationally invariant, as in ordinary relativistic QFT. This is obvious from the fact that the coproduct of the translation generators is not deformed, consequently the translations will act in an ordinary manner on the NC Wightman functions. However, there are again differences compared to the relativistic case. In relativistic QFT, the differences of coordinates on which the ordinary Wightman functions depend can be written as a four vector, $\xi_i^\mu = x_i^\mu - x_{i+1}^\mu$, since the functional form in all four coordinates is the same, due to Lorentz symmetry. In the noncommutative case, the Heisenberg fields $\phi_\theta(x)$ depend on the θ -matrix, and the same is valid for the \star -product of fields entering the NC Wightman functions. The Wightman functions are covariant only under the stability group $O(1,1) \times SO(2)$. Consequently, after shifting the coordinates by x_1 , the NC Wightman functions will depend with different functional forms on the two-dimensional vectors, $\vec{\sigma}_i = (\xi_i^0, \xi_i^1)$ and $\vec{\tau}_i = (\xi_i^2, \xi_i^3)$,

$$W_\star(x_1, x_2, \dots, x_n) = \mathcal{W}_\star(\vec{\sigma}_1, \vec{\tau}_1, \vec{\sigma}_2, \vec{\tau}_2, \dots, \vec{\sigma}_{n-1}, \vec{\tau}_{n-1}). \quad (2.24)$$

The subscript \star in the r.h.s. of (2.24) shows that the θ -dependence of the Wightman functions is not effaced by their translational invariance.

The translational invariance of NC Wightman functions as in (2.22) was used as argument in [21] for the disappearance of the \star -product upon translating by ξ , since the \star -product of the four-dimensional shifts ξ_i with any function of x reduces to the usual product. We should, however, be aware, that a function

$$F(x - y), \quad x, y \in \mathbf{R}^{1,3} \quad (2.25)$$

is translationally invariant, but so is also the function

$$F_0(x^0 - y^0, \theta') F_1(x^1 - y^1, \theta') F_2(x^2 - y^2, \theta) F_3(x^3 - y^3, \theta), \quad (2.26)$$

with the notation used in (2.15). Indeed, the \star -multiplication of two functions of the type (2.25) is the same like their usual multiplication, but the \star -product of two functions of the type (2.26) will retain its θ -dependence. The translational symmetry in the relativistically invariant case falls into the first situation, while in the noncommutative twisted Poincaré case, it falls into the latter.

2.5 Microcausality and spectral condition

In order to be able to define a microcausality condition, we should keep the locality in time, and therefore we shall restrict ourselves to theories with space-space noncommutativity. We

consider henceforth $\theta' = 0$ in (2.15). This situation corresponds to a $O(1, 1) \times \mathcal{T}_{(1,1)} = \mathcal{P}(1, 1)$ symmetry for the (x_0, x_1) plane and a $SO(2) \times \mathcal{T}_2 = E_2$ symmetry for the (x_2, x_3) plane. According to the present wisdom, the postulate of local commutativity is modified to require the vanishing of star-commutators of scalar fields at space-like separation in the sense of $O(1, 1)$ (i.e. replace light-cone by light-wedge):

$$[\phi_\theta(x), \phi_\theta(y)]_* \equiv \phi_\theta(x) \star \phi_\theta(y) - \phi_\theta(y) \star \phi_\theta(x) = 0, \\ \text{for } (x^0 - y^0)^2 - (x^1 - y^1)^2 < 0. \quad (2.27)$$

The lower symmetry compels us to enlarge, correspondingly, the physical spectrum of the momentum operator, i.e. the spectral condition, to the forward light-wedge:

$$\text{Spec}(p) = \{(p^0)^2 - (p^1)^2 \geq 0, p^0 \geq 0\}. \quad (2.28)$$

2.6 Space of test functions for noncommutative Wightman functions

Rigorously, field operators can not be defined at a point [24] (see also [1, 3]), but only as smoothed operators, written symbolically as

$$\varphi_f \equiv \int \varphi(x) f(x) dx, \quad (2.29)$$

where $f(x)$ are test functions.

In QFT, the standard assumption is that all $f(x)$ are test functions of tempered distributions. On the contrary, in the NC QFT the corresponding generalized functions can not be tempered distributions as the \star -product contains infinite number of derivatives. It is well known (see, for example, [1]) that there can be only a finite number of derivatives in any tempered distribution.

In [15] it was shown that the series

$$f(x) \star f(y) = \exp \left(\frac{i}{2} \theta^{\mu\nu} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial y^\nu} \right) f(x) f(y) \quad (2.30)$$

converges if $f(x) \in S^\beta$, $\beta < 1/2$, where S^β is a Gel'fand-Shilov space [22]. A similar result was obtained also in [23].

Thus the formal expression (2.22) actually means that the scalar product of the vectors $\Phi_k = \varphi_{f_k} \cdots \varphi_{f_1} |0\rangle$ and $\Psi_n = \varphi_{f_{k+1}} \cdots \varphi_{f_n} |0\rangle$ is the following:

$$\langle \Phi_k, \Psi_n \rangle = \int W(x_1, \dots, x_n) \overline{f_1(x_1)} \star \cdots \star \overline{f_k(x_k)} \star f_{k+1}(x_{k+1}) \star \cdots \star f_n(x_n) dx_1 \dots dx_n, \\ W(x_1, \dots, x_n) = \langle 0 | \varphi_\theta(x_1) \dots \varphi_\theta(x_n) | 0 \rangle. \quad (2.31)$$

Let us stress that after integration over the noncommutative variables $W(x_1, \dots, x_n)$ becomes tempered distribution with respect to commutative variables x_i^0, x_i^1 , $i = 1, 2, \dots, n$.

On the ground of this formulation we shall prove the CPT theorem for theories with space-space noncommutativity and also the spin-statistics theorem for the simplest case of scalar field.

We can use the formal expression (2.22) instead of the rigorous one (2.31). The point is that in accordance with the spectral property, the Wightman functions are analytical

functions with respect to the commutative coordinates. This property is crucial in our proof of the CPT and spin-statistics theorems. Let us point out that CPT theorem and spin statistic one can be proved under conditions different from ours, namely by using the concept of asymptotic commutativity condition [25]. We assume that Wightman functions are tempered distributions in respect with commutative coordinates as this natural physics assumption gives us the possibility to prove in NC QFT the main axiomatic results of ordinary quantum field theory such as irreducibility of the set of quantum field operators and generalized Haag's theorem [26].

3 CPT theorem in the axiomatic approach to NC QFT

In ordinary QFT, CPT theorem was proven in the Lagrangian approach, by Lüders and Pauli [27]. A general proof was given by Jost in the axiomatic formulation [28] (see also [1]). In the NC case, CPT theorem was shown to hold in NC QED [29]. A general proof in the Lagrangian formalism, for any NC QFT, was given in [9].

In the axiomatic approach to the ordinary QFT, the CPT theorem states [1] that the CPT invariance condition in terms of Wightman functions, e.g. in the case of a neutral scalar field, i.e.

$$W(x_1, x_2, \dots, x_n) = W(-x_n, \dots, -x_2, -x_1), \quad (3.32)$$

for any values of x_1, x_2, \dots, x_n , is equivalent to the weak local commutativity (WLC) condition,

$$W(x_1, x_2, \dots, x_n) = W(x_n, \dots, x_2, x_1), \quad (3.33)$$

where $x_1 - x_2, \dots, x_{n-1} - x_n$ is a Jost point, i.e. it satisfies the condition that

$$\left(\sum_{j=1}^{n-1} \lambda_j (x_j - x_{j+1}) \right)^2 < 0, \quad \text{for all } \lambda_j \geq 0 \text{ with } \sum_{j=1}^{n-1} \lambda_j > 0.$$

The main ingredients for the proof are the analyticity of the Wightman functions and the fact that the space-time inversion (PT-transformation) is connected to the identity in the complex Lorentz group.

3.1 CPT invariance and WLC conditions in terms of the NC Wightman functions

In order to prove the CPT theorem in NC QFT, the first of our concerns is to derive the CPT invariance and !!!WLC conditions!!! in terms of the new Wightman functions defined by eq. (2.22).

In the following, for simplicity, we shall restrict ourselves to the case of one neutral scalar field.

The CPT invariance condition is derived by requiring that the CPT operator Θ be antiunitary (see, e.g. [1]):

$$\langle \Theta\Phi | \Theta\Psi \rangle = \langle \Psi | \Phi \rangle, \quad (3.34)$$

i.e. the CPT operator leaves invariant all transition probabilities of the theory.

Taking the vector states as $\langle \Phi | = \langle 0 | \equiv \langle \Psi_0 |$ and $|\Psi\rangle = \phi_\theta(x_n) \star \dots \star \phi_\theta(x_2) \star \phi_\theta(x_1) |\Psi_0\rangle$ we shall express both sides of (3.34) in terms of NC Wightman functions.

For the l.h.s. of (3.34) we use directly the CPT transformation properties of the field operators, which read, for a neutral scalar field, $\Theta\phi_\theta(x)\Theta^{-1} = \phi_\theta(-x)$. Using the CPT-invariance of the vacuum state, $\Theta|\Psi_0\rangle = |\Psi_0\rangle \equiv |0\rangle$, the l.h.s. of (3.34) becomes:

$$\begin{aligned}\langle \Theta\Phi | \Theta\Psi \rangle &= \langle \Theta\Psi_0 | \Theta(\phi_\theta(x_n) \star \dots \star \phi_\theta(x_2) \star \phi_\theta(x_1)) |\Psi_0\rangle \\ &= W_\star(-x_n, \dots, -x_2, -x_1).\end{aligned}\quad (3.35)$$

For expressing the r.h.s. of (3.34) we take the hermitian conjugates of the vectors $|\Psi\rangle$ and $\langle \Phi |$, to obtain:

$$\langle \Psi | \Phi \rangle = W_\star(x_1, x_2, \dots, x_n). \quad (3.36)$$

Putting together (3.34) with (3.35) and (3.36), we obtain the CPT invariance condition in terms of NC Wightman functions as

$$W_\star(x_1, x_2, \dots, x_n) = W_\star(-x_n, \dots, -x_2, -x_1). \quad (3.37)$$

Let us introduce the WLC condition. We remark that the \star -products contained in the definition of the Wightman functions do not influence in any way the coordinates involved in defining the light-wedge in (2.27), i.e. x^0 and x^1 . Consequently, at space-like separated points in the sense of $SO(1, 1)$ (denoted by $x_i \sim x_j$, $i, j = 1, 2, \dots, n$), we can permute the field operators in (2.22) in accordance with (2.27). The WLC condition implies only that Wightman functions are not changed if direct order of points x_i is substituted by the inverse one. Thus the noncommutative version of the WLC condition in terms of Wightman functions reads:

$$\begin{aligned}W_\star(x_1, x_2, \dots, x_n) &= W_\star(x_n, \dots, x_2, x_1) \\ \text{for } x_i &\sim x_j, \quad i, j = 1, 2, \dots, n.\end{aligned}\quad (3.38)$$

Finally the proof of the CPT theorem amounts to showing the equivalence of (3.37) and (3.38).

3.2 Proof of CPT theorem

The CPT theorem consists of the equivalence of (3.37) and (3.38). The proof goes along similar lines as in the commutative case [1] or the case discussed in [12]. The main step is the analytical continuation of the Wightman functions to the complex plane only with respect to x_0 and x_1 . The \star -products introduced in the new Wightman functions have no influence on these two coordinates (since we have taken $\theta_{01} = 0$, i.e. the time to be commutative, cf. (2.15)) and upon analytical continuation they will not be affected. Due to the translational invariance, we can express $W_\star(x_1, x_2, \dots, x_n)$ in terms of the $2(n-1)$ relative variables, $\vec{\sigma}_i = (\xi_i^0, \xi_i^1)$ and $\vec{\tau}_i = (\xi_i^2, \xi_i^3)$, with $\xi_i^\mu = x_i^\mu - x_{i+1}^\mu$:

$$W_\star(x_1, x_2, \dots, x_n) = \mathcal{W}_\star(\vec{\sigma}_1, \dots, \vec{\sigma}_{n-1}, \vec{\tau}_1, \dots, \vec{\tau}_{n-1}) \quad (3.39)$$

(see the discussion in Subsection 2.4).

From (2.28), the spectral condition for NC Wightman functions follows. Specifically, the Fourier transform of a Wightman function is nonzero, i.e.

$$\tilde{\mathcal{W}}_\star(\vec{\mathcal{P}}_1, \vec{\mathcal{P}}_2, \dots, \vec{\mathcal{P}}_{n-1}) = \frac{1}{(2\pi)^{2(n-1)}} \int \Pi_{i=1}^{n-1} d\vec{\sigma}_k$$

$$\times e^{-i(\mathcal{P}_k^0\sigma_k^0 - \mathcal{P}_k^1\sigma_k^1)} \mathcal{W}_*(\vec{\sigma}_1, \dots, \vec{\sigma}_{n-1}, \vec{\tau}_1, \dots, \vec{\tau}_{n-1}) \neq 0. \quad (3.40)$$

only if all the two-dimensional momenta $\vec{\mathcal{P}}_i = (\mathcal{P}_i^0, \mathcal{P}_i^1)$ satisfy the conditions (2.28). In (3.40) the inessential dependence of the l.h.s. on the vectors $\vec{\tau}_i$ is omitted. The condition (3.40) implies that, on the same grounds as in commutative case [1], NC Wightman functions can be analytically continued into the complex plane with respect to ξ_i^0 and ξ_i^1 , by the substitution $\xi_i \rightarrow \mu_i = \xi_i - i\eta_i$, with all η_i belonging to the forward light-wedge, i.e. $(\eta_i^0)^2 - (\eta_i^1)^2 \geq 0$, $\eta_i^0 \geq 0$ and $\eta_i^2 = \eta_i^3 = 0$. In other words,

$$\begin{aligned} \mathcal{W}_*(\mu_1, \mu_2, \dots, \mu_{n-1}) &= \frac{1}{(2\pi)^{2(n-1)}} \int \prod_{k=1}^{n-1} d\vec{\mathcal{P}}_k \\ &\times e^{i(\mathcal{P}_k^0\mu_k^0 - \mathcal{P}_k^1\mu_k^1)} \tilde{\mathcal{W}}_*(\vec{\mathcal{P}}_1, \vec{\mathcal{P}}_2, \dots, \vec{\mathcal{P}}_{n-1}) \end{aligned} \quad (3.41)$$

is analytical in μ_i^0 and μ_i^1 . In accordance with the Bargmann-Hall-Wightman theorem [1, 30], the functions $\mathcal{W}_*(\mu_1, \mu_2, \dots, \mu_{n-1})$ are analytical in an extended domain; in commutative case the extended domain is obtained from the initial one by applying all (proper) complex Lorentz transformations, continuously related to the unit transformation. In the noncommutative case we can use only the transformations belonging to the complex $O(1, 1)$ [12] for obtaining the extended domain of analyticity. This domain contains also the real points $\tilde{\xi}_i$ called Jost points, with the property that $\left(\sum_{j=1}^{n-1} \lambda_j \tilde{\xi}_j^0\right)^2 - \left(\sum_{j=1}^{n-1} \lambda_j \tilde{\xi}_j^1\right)^2 < 0$, for all $\lambda_j \geq 0$ with $\sum \lambda_j > 0$ (consequently, the points x_1, x_2, \dots, x_n are mutually space-like in the sense of $O(1, 1)$). The values of Wightman functions at Jost points fully determine the values in the whole domain of analyticity. Thus, two Wightman functions coinciding at their Jost points coincide everywhere.

We shall show now that the WLC condition (3.38) implies the CPT invariance condition (3.37). Let us first rewrite (3.38) in terms of relative coordinates:

$$\mathcal{W}_*(\tilde{\xi}_1, \tilde{\xi}_2, \dots, \tilde{\xi}_{n-1}) = \mathcal{W}_*(-\tilde{\xi}_{n-1}, \dots, -\tilde{\xi}_2, -\tilde{\xi}_1). \quad (3.42)$$

The functions $\mathcal{W}_*(\xi_1, \dots, \xi_{n-1})$ and $\mathcal{W}_*(-\xi_{n-1}, \dots, -\xi_1)$ satisfy the spectral condition (3.40) and are invariant under $O(1, 1)$ transformations. Thus, in accordance with the previous arguments, they are both analytical functions of the complex variables μ_i in the above-mentioned extended domain. Moreover, since they are equal at the Jost points, they are also equal in the whole domain of analyticity. Using the invariance of $\mathcal{W}_*(\mu_1, \mu_2, \dots, \mu_{n-1})$ and $\mathcal{W}_*(-\mu_{n-1}, \dots, -\mu_2, -\mu_1)$ under the complex $O(1, 1)$ group, which includes the inversion $\mu_i^0 \rightarrow -\mu_i^0$ and $\mu_i^1 \rightarrow -\mu_i^1$, we arrive at the equality

$$\mathcal{W}_*(\mu_1, \dots, \mu_{n-1}) = \mathcal{W}_*(-\mu'_{n-1}, \dots, -\mu'_1), \quad (3.43)$$

where $\mu'_i = (-\mu_i^0, -\mu_i^1, \mu_i^2, \mu_i^3) \equiv (-\mu_i^0, -\mu_i^1, \tau_i^2, \tau_i^3)$. Performing a $SO(2)$ rotation by π in the (τ^2, τ^3) plane and subsequently going to the real limit, we obtain that

$$\mathcal{W}_*(\xi_1, \xi_2, \dots, \xi_{n-1}) = \mathcal{W}_*(\xi_{n-1}, \dots, \xi_2, \xi_1), \quad (3.44)$$

which is equivalent to the CPT invariance condition (3.37) in terms of x_1, x_2, \dots, x_n . Thus CPT invariance is the consequence of WLC. By similar considerations the converse can also be proven, i.e. CPT invariance implies WLC.

4 Spin-statistics theorem

Within the Lagrangian framework the spin-statistics theorem has been shown [9] to hold for NC QFT with space-space noncommutativity. The proof used Pauli's original formulation [31], requiring that at space-like separations the commutators of two observables should vanish.

Moreover, the symmetry under twisted Poincaré algebra gives an additional hint that the spin-statistics relation should survive in the NC QFT. In the case of a usual Lie algebra, the operator of permutation P inverts the order of two representations in a tensor product: $P(a \otimes b) = b \otimes a$. In the case of a quantum group with a universal enveloping algebra \mathcal{R} , the notion of permutation operator changes, such that the new permutation operator, Ψ , is consistent with the quantum group action, i.e. $\Delta(Y)(\Psi(a \otimes b)) = \Psi(\Delta(Y)(a \otimes b))$, where $\Delta(Y)$ is the deformed coproduct of a Lie algebra generator. Using (2.19), it follows that $\Psi(a \otimes b) = P(\mathcal{R}(a \otimes b))$. The twisted Poincaré algebra is a particular case of quantum group, called triangular Hopf algebra, for which $\mathcal{R}^{-1} = \mathcal{R}_{21}$. Consequently, $\Psi = \Psi^{-1}$, i.e. Ψ is symmetric and no exotic statistics emerges, but the representations of the twisted Poincaré algebra will have the same statistics as those of the ordinary Poincaré symmetry (see [17]). These aspects refer to the spin-statistics *relation* for particle representations, and not to quantum field theory. Similar considerations based on the twisted \mathbb{P} -symmetry can be found as well in [32].

Encouraged by these indications, we attempt here to show the validity of the spin-statistics theorem based on the NC Wightman functions, for the case of a real scalar field.

We start by proving that, if $\psi_\theta(x)|0\rangle = 0$ and $\psi_\theta(x)$ is a local field operator in the sense of the light-wedge, then $\psi_\theta(x) = 0$. To show this, we take at the Jost points $\tilde{x}_1 - \tilde{x}_2, \dots, \tilde{x}_j - \tilde{x}, \tilde{x} - \tilde{x}_{j+1}, \dots, \tilde{x}_{n-1} - \tilde{x}_n$ the arbitrary NC Wightman function

$$\begin{aligned} & \langle 0 | \phi_\theta(\tilde{x}_1) \star \dots \star \phi_\theta(\tilde{x}_j) \star \psi_\theta(\tilde{x}) \star \phi_\theta(\tilde{x}_{j+1}) \dots \star \phi_\theta(\tilde{x}_n) | 0 \rangle \\ &= \langle 0 | \phi_\theta(\tilde{x}_1) \dots \phi_\theta(\tilde{x}_j) \star \phi_\theta(\tilde{x}_{j+1}) \dots \phi_\theta(\tilde{x}_n) \star \psi_\theta(\tilde{x}) | 0 \rangle = 0 . \end{aligned} \quad (4.45)$$

By analytically continuing the first line of (4.45) (in exactly the same way as for the proof of the CPT theorem in the previous section), we obtain $\langle 0 | \phi_\theta(x_1) \star \dots \star \phi_\theta(x_j) \star \psi_\theta(x) \star \phi_\theta(x_{j+1}) \dots \star \phi_\theta(x_n) | 0 \rangle = 0$, i.e. all the matrix elements of the operator $\psi_\theta(x)$ between a complete set of states $\langle 0 | \phi_\theta(x_1) \star \dots \star \phi_\theta(x_j)$ and $\phi_\theta(x_{j+1}) \dots \star \phi_\theta(x_n) | 0 \rangle$ are zero and thus $\psi_\theta(x) = 0$.

Now we show that the wrong statistics,

$$\{\phi_\theta(x), \phi_\theta(y)\}_\star = 0, \quad (x^0 - y^0)^2 - (x^1 - y^1)^2 < 0 , \quad (4.46)$$

leads to $\phi_\theta(x) = 0$.

Consider $W_\star(x, y) = \langle 0 | \phi_\theta(x) \star \phi_\theta(y) | 0 \rangle$. According to (4.46) we have

$$W_\star(\tilde{x}, \tilde{y}) + W_\star(\tilde{y}, \tilde{x}) = 0 . \quad (4.47)$$

Eq. (4.47) can be analytically continued, as in the previous section, into the extended domain. Performing a space-time inversion and taking the real limit for the coordinates, we obtain for the second term of (4.47) $W_\star(y, x) = W_\star(x, y)$. Thus, $W_\star(x, y) = 0$. At $y = x$ we get

$$\langle 0 | \phi_\theta(x) \star \phi_\theta(x) | 0 \rangle = 0 , \quad (4.48)$$

which is equivalent to $\langle \Psi | \Psi \rangle = 0$, with $|\Psi\rangle = \phi_\theta(x)|0\rangle$, if one adopts the definition for the norm of a state as in (4.48), or equivalently as $\langle \Psi | \Psi \rangle = \langle 0 | \phi_\theta(\hat{x}) \phi_\theta(\hat{x}) | 0 \rangle$. Then $\phi_\theta(x)|0\rangle = 0$ and, due to the result first derived, we get $\phi_\theta(x) = 0$.

5 Discussion and conclusions

In this paper we have introduced new Wightman functions defined as vacuum expectation values of the product of field operators with \star -product. In this attempt to develop an axiomatic formulation of NC QFT, the noncommutativity of space-time is explicitly taken into account. Using such NC Wightman functions, for the case of only space-space noncommutativity, $\theta_{0i} = 0$, we prove the CPT theorem adapted to the symmetry $\mathcal{P}(1,1) \times E_2$. From this and the previous study made in [12], one arrives at the conclusion that in a theory with violation of CPT theorem, not only the Lorentz invariance is violated, but also its subgroup of symmetry $O(1,1) \times SO(2)$ will be broken*. This result is of fundamental importance for experimentally detecting a possible violation of Lorentz invariance through a CPT violation effect.

For the case of a scalar field we also prove the spin-statistics theorem using the NC Wightman functions with $\theta_{0i} = 0$. The causality condition used is the vanishing of star-commutator of the scalar field at light-wedge space-like points, as in eq. (2.27).

The twisted Poincaré symmetry of NC QFT is crucial for the accurate axiomatic formulation: *i*) it justifies the use of the \star -product in the construction of the Wightman functions (see section 2.3) and *ii*) it allows us altogether to speak rigorously about the concept of spin in NC QFT (a concept which would be nonexistent in the residual symmetry formulation) and thus to pose the problem of the spin-statistics relation (see section 2.1 and Ref. [13]). For the actual proof of the CPT and spin-statistics theorem, the interplay of the twisted Poincaré and the residual $O(1,1) \times SO(2)$ symmetry is essential (see section 2.2 and Refs. [18, 19]). The analytical properties of the smeared new Wightman functions can be utilized in the proofs due to the existence of the corresponding class of test functions, namely the Gelfand-Shilov space S^β , $\beta < 1/2$ (see section 2.6 and Ref. [15] for details).

There are still several important questions to be studied within the axiomatic formulation of NC QFT, e.g. the cluster decomposition property, the uniqueness of the vacuum state related to the irreducibility of field operator algebra, the proof of the analytic properties of the scattering amplitude [34] and their implication on its high-energy behaviour [35]. Finally, among the most important problems remains the formulation and proof of the reconstruction theorem. We hope to present further results on these problems in a future communication.

Acknowledgements We are much grateful to L. Álvarez-Gaumé and C. Montonen for useful discussions. The support of the Academy of Finland under the Projects No. 121720 and 127626 is greatly acknowledged.

References

- [1] R. F. Streater and A. S. Wightman, *PCT, Spin, Statistics and All That*, W. A. Benjamin, Inc., New York, 1964, and references therein.
- [2] R. Jost, *The General Theory of Quantized Fields*, American Mathematical Society, Providence, 1965.

*O. W. Greenberg has shown, by introducing different masses for a particle and its antiparticle, that the violation of CPT invariance implies the breaking of the Lorentz symmetry [33]. The results of the present paper and those obtained in [12] can be considered as a general proof from which, in addition, the violation of $O(1,1) \times SO(2)$ follows.

- [3] N. N. Bogoliubov, A. A. Logunov and I. T. Todorov, *Introduction to Axiomatic Quantum Field Theory*, W. A. Benjamin, Inc., New York, 1975.
- [4] R. Haag, *Local Quantum Physics*, Springer, Berlin, 1996.
- [5] M. Froissart, *Asymptotic Behavior and Subtractions in the Mandelstam Representation*, *Phys. Rev.* **123**, 1053 (1961);
A. Martin, *Extension of the axiomatic analyticity domain of scattering amplitudes by unitarity. 1.*, *Nuovo Cim.* **42**, 930 (1966).
- [6] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *JHEP* **09**, 32 (1999) [arXiv:hep-th/9908142].
- [7] J. Gomis and T. Mehen, *Space-time noncommutative field theories and unitarity*, *Nucl. Phys. B* **591**, 265 (2000) [arXiv:hep-th/0005129].
- [8] N. Seiberg, L. Susskind and N. Toumbas, *Space-time noncommutativity and causality*, *JHEP* **06**, 044 (2000) [arXiv:hep-th/0005015];
L. Álvarez-Gaumé and J. L. F. Barbon, *Nonlinear vacuum phenomena in noncommutative QED*, *Int. J. Mod. Phys A* **16**, 1123 (2001) [arXiv:hep-th/0006209].
- [9] M. Chaichian, K. Nishijima and A. Tureanu, *Spin statistics and CPT theorems in non-commutative field theory*, *Phys. Lett. B* **568**, 146 (2003) [arXiv:hep-th/0209006].
- [10] M. R. Douglas and N. A. Nekrasov, *Noncommutative field theory*, *Rev. Mod. Phys.* **73**, 977 (2001) [arXiv:hep-th/0106048].
- [11] R. J. Szabo, *Quantum field theory on noncommutative spaces*, *Phys. Rept.* **378**, 207 (2003) [arXiv:hep-th/0109162].
- [12] L. Álvarez-Gaumé and M. A. Vázquez-Mozo, *General properties of noncommutative field theories*, *Nucl. Phys. B* **668**, 293 (2003) [arXiv:hep-th/0305093].
- [13] M. Chaichian, P. Kulish, K. Nishijima and A. Tureanu, *On a Lorentz-invariant interpretation of noncommutative space-time and its implications on noncommutative QFT*, *Phys. Lett. B* **604**, 98 (2004) [arXiv:hep-th/0408069].
- [14] M. Chaichian, P. Prešnajder and A. Tureanu, *New concept of relativistic invariance in NC space-time: Twisted Poincaré symmetry and its implications*, *Phys. Rev. Lett.* **94**, 151602 (2005) [arXiv:hep-th/0409096].
- [15] M. Chaichian, M. N. Mnatsakanova, A. Tureanu and Yu. S. Vernov, *Test Functions Space in Noncommutative Quantum Field Theory*, *JHEP* **09**, 125 (2008) [arXiv:0706.1712 [hep-th]].
- [16] V. G. Drinfel'd, *Quasi-Hopf algebras*, *Leningrad Math. J.* **1**, 1419 (1990).
- [17] V. Chari and A. Pressley, *A Guide to Quantum Groups*, Cambridge University Press, Cambridge, 1994;
S. Majid, *Foundations of Quantum Group Theory*, Cambridge University Press, Cambridge, 1995;
M. Chaichian and A. Demichev, *Introduction to Quantum Groups*, World Scientific, Singapore, 1996.
- [18] M. Chaichian, P. P. Kulish, A. Tureanu, R. B. Zhang and Xiao Zhang, *Noncommutative fields and actions of twisted Poincaré algebra*, *J. Math. Phys.* **49**, 042302 (2008) [arXiv:0711.0371].

- [19] M. Chaichian, K. Nishijima, T. Salminen and A. Tureanu, *Noncommutative Quantum Field Theory: A Confrontation of Symmetries*, *JHEP* **06**, 078 (2008) [arXiv:0805.3500].
- [20] M. Chaichian, K. Nishijima and A. Tureanu, *An Interpretation of noncommutative field theory in terms of a quantum shift*, *Phys. Lett.* **B 633**, 129 (2006) [arXiv:hep-th/0511094].
- [21] G. Fiore and J. Wess, *On full twisted Poincaré symmetry and QFT on Moyal-Weyl spaces*, *Phys. Rev.* **D 75**, 105022 (2007) [arXiv:hep-th/0701078].
- [22] I. M. Gel'fand and G. E. Shilov, *Generalized Functions*, vol. 2, chapter IV, Academic Press Inc., New York (1968).
- [23] M. A. Soloviev, *Noncommutativity and theta-locality*, *J. Phys.* **A 40**, 14593 (2007) [arXiv:0708.0811[hep-th]].
- [24] A. S. Wightman, *Quantum Field Theory in Terms of Vacuum Expectation Values*, *Phys. Rev.* **101**, 860 (1956).
- [25] M. A. Soloviev, *Axiomatic formulations of nonlocal and noncommutative field theories*, *Theor. Math. Phys.* **147**, 660 (2006), *Teor. Mat. Fiz.* **147**, 257 (2006), [arXiv:hep-th/0605249].
- [26] M. Chaichian, M. N. Mnatsakanova, A. Tureanu and Yu. S. Vernov, *Classical Theorems in Noncommutative Quantum Field Theory*, hep-th/0612112.
- [27] G. Lüders, *On the Equivalence of Invariance under Time-Reversal and under Particle-Antiparticle Conjugation for Relativistic Field Theories*, *Dan. Mat. Fys. Medd.* **28**, 5 (1954);
W. Pauli, *Niels Bohr and the Development of Physics*, W. Pauli (ed.), Pergamon Press, New York, 1955.
- [28] R. Jost, *A remark on the C.T.P. theorem*, *Helv. Phys. Acta* **30**, 409 (1957); *Theoretical Physics in the Twentieth Century*, Interscience, New York, 1960.
- [29] M. M. Sheikh-Jabbari, *C, P, and T invariance of noncommutative gauge theories*, *Phys. Rev. Lett.* **84**, 5265 (2000) [arXiv:hep-th/0001167].
- [30] D. W. Hall and A. S. Wightman, *A Theorem on Invariant Analytic Functions with Applications to Relativistic Quantum Field Theory*, *Mat. Fys. Medd. Dan. Vid. Selsk.* **31**, 5 (1957).
- [31] W. Pauli, *The Connection Between Spin and Statistics*, *Phys. Rev.* **58**, 716 (1940); *Progr. Theor. Phys. (Kyoto)* **5**, 526 (1950).
- [32] A. Tureanu, *Twist and spin-statistics relation in noncommutative quantum field theory*, *Phys. Lett.* **B 638**, 296 (2006) [arXiv:hep-th/0603219]; *Twisted Poincaré symmetry and some implications on noncommutative quantum field theory*, *Prog. Theor. Phys. Suppl.* **171**, 34 (2007) [arXiv:0706.0334].
- [33] O. W. Greenberg, *CPT violation implies violation of Lorentz invariance*, *Phys. Rev. Lett.* **89**, 231602 (2002) [arXiv:hep-ph/0201258].
- [34] Y. Liao and K. Sibold, *Spectral representation and dispersion relations in field theory on noncommutative space*, *Phys. Lett.* **B 549**, 352 (2002) [arXiv:hep-th/0209221];
M. Chaichian, M. N. Mnatsakanova, A. Tureanu and Yu. S. Vernov, *Analyticity and forward dispersion relations in noncommutative quantum field theory*, *Nucl. Phys.* **B 673**, 476 (2003) [arXiv:hep-th/0306158].

- [35] M. Chaichian and A. Tureanu, *Jost-Lehmann-Dyson representation and Froissart-Martin bound in quantum field theory on noncommutative space-time* [arXiv:hep-th/0403032];
A. Tureanu, *Analyticity of the scattering amplitude, causality and high-energy bounds in quantum field theory on noncommutative space-time*, *J. Math. Phys.* **47**, 092302 (2006) [arXiv:hep-th/0603029].